Suggested solution of HW6

Q1 Following the hints. It suffices to show (2). Let (a_n, b_n) be the open interval in F^c such that $x \in (a_n, b_n)$. Then a_n cannot be smaller than x_0, otherwise $x_0 \in F^c$. Similarily, $b_n \leq x_0 + \delta$. Since $a_n, b_n \in F$,

$$|g(b_n) - g(x_0)|, |g(a_n) - g(x_0)| < \epsilon.$$

And hence, for $x = ta_n + (1-t)b_n$,

$$g(x) = tg(a_n) + (1-t)g(b_n) \in \big(g(x_0) - \epsilon, g(x_0) + \epsilon\big).$$

Q2 Clearly, φ_n is a simple function and hence measurable. It suffices to check the monotone nature. If $f(x) \geq n+1$, then $\varphi_{n+1}(x) = n+1 \geq \varphi_n(x)$.

If $n+1 > f(x) \geq n = n \cdot \frac{2^{n+1}}{2^{n+1}}$, then $\varphi_{n+1}(x) \geq n$.

If $\frac{k-1}{2^n} \leq f(x) < \frac{k}{2^n}$, then either $\frac{2k-1}{2^{n+1}} \leq f(x) < \frac{2k}{2^{1+n}}$ or $\frac{2k-2}{2^{n+1}} \leq f(x) < \frac{2k-1}{2^{1+n}}$. In both cases, $\varphi_{n+1} \geq \varphi_n$. Hence φ_n is monotone. If $f(x)$ is finite, then

$$|f(x) - \varphi_n(x)| \leq \frac{1}{2^n} \to 0.$$

If $f(x) = +\infty$, then $\varphi_n(x) = n \to f(x)$.

Q3 Rewrite $f(\lambda x) = f \circ g(x)$. Since

$$g^{-1} \circ f^{-1}(\alpha, +\infty) = \{f(g(x)) > \alpha\}.$$

It suffices to show that linear function F map measurable set to measurable set. By inner regularity and taking intersection with $[-n, n]$, we may assume

$$E = \cup E_n \cup N$$

where E_n is compact and N is null. Since F is continuous function, $F(E_n)$ is compact and hence measurable. It suffices to show that $F(N)$ is measurable. Let $\epsilon > 0$, there is $\{I_n\}_{n=1}^\infty$ such that $N \subset \bigcup I_n$ and $\sum_{n=1}^\infty \ell(I_n) < \epsilon$. Write $I_n = (a_n, b_n)$. Then $|F(a_n) - F(b_n)| \leq C|a_n - b_n| = C\ell(I_n)$. Therefore,

$$F(N) \leq \mu(F(\cup I_n)) \leq \sum_{n=1}^\infty \mu(F(I_n)) \leq C\epsilon.$$

Hence, $F(N)$ is null and thus measurable. Put $g = x + c$ and $g(x) = \lambda x$ to conclude the results.